Boundedness and Blowup for Nonlinear Degenerate Parabolic Equations

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Abstract

The author deals with the quasilinear parabolic equation $u_t = [u^\alpha + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u)$ with Dirichlet boundary conditions in a bounded domain Ω , where f and g are lower order terms. He shows that, under suitable conditions on f and g , whether the solution is bounded or blows up in a finite time depends only on the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition. For some special cases, the result is sharp.

Key words: porous medium equation; quasilinear parabolic equation; global existence; blowup solutions.

In this paper we consider the following quasilinear parabolic equation:

$$u_t = [u^\alpha + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u), \quad t > 0, \quad x \in \Omega, \quad (1)$$

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with the initial and boundary conditions

$$\begin{cases} u(x, t) = 0, & t > 0, & x \in \partial\Omega, \\ u(x, 0) = \phi(x) > 0, & x \in \Omega, \end{cases} \quad (2)$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$.

The study of (1) and (2) is of great interest due to its applications. If $g(0) = 0$, (1) is the well-known porous medium equation with a source and a convection term (see [1], [5] [7], [8], [18], [20] and [22]). In the simple cases where $g = f = 0$, many authors (see [2], [15] and [22]) proved that whether the solution exists globally or blows up in a finite time depends only on the first eigenvalue of $-\Delta$ in the domain Ω with Dirichlet boundary condition.

Ding *et al* [11] and [12] discussed the general quasilinear parabolic equations and obtained some interesting blowup results. Suzuki [23] considered the following equation

$$u_t = \Delta u^m + a \cdot \nabla u^q + u^p,$$

where $1 \leq m < p < m + 2/(N + 1)$ and $(m + 1)/2 \leq q < (m + p)/2$. He obtained a universal bound which is independent of the initial data for all global nonnegative solutions of the Dirichlet problem. Chen and Yu [5] dealt with the following equation

$$u_t = \nabla \cdot (g(u)\nabla u) + h(u, \nabla u) + f(u)$$

with $u|_{\partial\Omega} = 0$, $u(x, 0) = \phi(x)$. If f, g and h are polynomials with proper degrees and proper coefficients, they showed that the blowup property depends only on the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition. For a special case, they obtain a sharp result. Many authors also discussed the blowup properties of local or nonlocal sources or degenerate parabolic systems and obtained similar interesting results (see [6], [9], [10], [13], [14], [16],

[19] and [21]).

In this paper, we use a new method to obtain both lower and upper bound for the solution of (1). In this method, we estimate the integral of a ratio of one solution to the other. This method shows successful in proving existence and blowup problems (see [3] and [5]), especially, the problems where the comparison principle is failed (see [4]). We first prove the boundedness of the solutions and then the blowup properties under different conditions on f and g for the problem (1) and (2). In some special cases, the result is sharp.

Throughout this paper, we denote by $\lambda_1 = \lambda_1(\Omega)$ the first eigenvalue of

$$\Delta\psi + \lambda_1\psi = 0 \quad \text{with} \quad \psi|_{\partial\Omega} = 0, \quad (3)$$

where ψ is the first normalized eigenfunction.

Theorem 1 *Suppose that, for any $\xi, \varepsilon > 0$ and $u > 0$, there exist positive constants $c = c(\varepsilon)$ and c_0 such that*

$$|f(u, \xi)| \leq \varepsilon u^{\alpha-1}(u^2 + |\xi|^2) + c(\varepsilon)u^{-\sigma}, \quad (4)$$

and

$$g(u) \geq c_0 u^\beta \quad \text{and} \quad g(u) \geq c_0 u g'(u) \geq 0, \quad (5)$$

where $\beta \geq 0$ and $\sigma \geq 0$. If $b < \lambda_1$, then, for any initial value $\phi(x) \in C_0^{1+\eta}(\overline{\Omega})$ with $\eta > 0$, the solution of the problem (1) and (2) is bounded for all $t > 0$.

Proof. By Lemma 1 in [5], the solution u is positive in Ω . This guarantees that u is a classical solution as long as u is bounded. By the continuity of the eigenvalues with respect to the domain Ω (see [17]), we can find a small $\delta > 0$ such that $b < \lambda_1 - \delta$ and we can define a domain $D \supset \Omega$ such that $\partial\Omega$ is located inside of D and there exists the first normalized

eigenfunction of

$$\Delta\psi_1 + (\lambda_1 - \delta)\psi_1 = 0 \quad (6)$$

in D with $\psi_1|_{\partial D} = 0$. Then ψ_1 is positive in $\bar{\Omega}$. For any positive number n , define

$$w_n(t) = \int_{\Omega} \frac{u^n(x, t)}{\psi_1^{kn}(x)} dx, \quad (7)$$

where $k < 1$ satisfying $b < \lambda_1 k - \delta$. Differentiating (7), substituting the equation of (1) into the result integral, integrating by parts and using (4), we have

$$\begin{aligned} \frac{d}{dt}w_n(t) &= n \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} \{[u^{\alpha} + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u)\} dx \\ &\leq -n(n-1+\alpha) \int_{\Omega} \frac{u^{n-2+\alpha}}{\psi_1^{kn}} |\nabla u|^2 dx + kn^2 \int_{\Omega} \frac{u^{n-1+\alpha}}{\psi_1^{kn+1}} \nabla u \nabla \psi_1 dx \\ &\quad - n(n-1) \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn}} g(u) |\nabla u|^2 dx + kn^2 \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g(u) \nabla u \nabla \psi_1 dx \\ &\quad - n \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx + nb \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn}} dx \\ &\quad + n \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} [\varepsilon(u^{\alpha+1} + u^{\alpha-1} |\nabla u|^2) + c(\varepsilon)u^{-\sigma}] dx. \end{aligned} \quad (8)$$

To simplify (8), we use the identity

$$\psi_1^2 |\nabla u|^2 = |\psi_1 \nabla u - ku \nabla \psi_1|^2 + 2ku \psi_1 \nabla u \nabla \psi_1 - k^2 u^2 |\nabla \psi_1|^2. \quad (9)$$

Then (8) becomes

$$\begin{aligned}
\frac{d}{dt}w_n(t) &\leq -n(n-1+\alpha-\varepsilon) \int_{\Omega} \frac{u^{n-2+\alpha}}{\psi_1^{kn+2}} |\psi_1 \nabla u - ku \nabla \psi_1|^2 dx \\
&\quad - n[kn - 2k(1-\alpha+\varepsilon)] \int_{\Omega} \frac{u^{n-1+\alpha}}{\psi_1^{kn+1}} \nabla u \nabla \psi_1 dx \\
&\quad + n(n-1+\alpha-\varepsilon)k^2 \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+2}} |\nabla \psi_1|^2 dx \\
&\quad - [n(n-1) - c_0^{-2}] \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn+2}} g(u) |\psi_1 \nabla u - ku \nabla \psi_1|^2 dx \\
&\quad - c_0^{-2} \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn}} g(u) |\nabla u|^2 dx - n \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx \\
&\quad - k[n(n-2) - 2c_0^{-2}] \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g(u) \nabla u \nabla \psi_1 dx + n(b+\varepsilon) \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn}} dx \\
&\quad + [n(n-1) - c_0^{-2}]k^2 \int_{\Omega} \frac{u^n}{\psi_1^{kn+2}} g(u) |\nabla \psi_1|^2 dx + nc(\varepsilon) \int_{\Omega} \frac{u^{n-1-\sigma}}{\psi_1^{kn}} dx. \tag{10}
\end{aligned}$$

Integrating by parts for the second and seventh terms in (10) and using (6), we obtain

$$\begin{aligned}
\int_{\Omega} \frac{u^{n-1+\alpha}}{\psi_1^{kn+1}} \nabla u \nabla \psi_1 dx &= \frac{kn+1}{n+\alpha} \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+2}} |\nabla \psi_1|^2 dx - \frac{1}{n+\alpha} \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+1}} \Delta \psi_1 dx \\
&= \frac{kn+1}{n+\alpha} \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+2}} |\nabla \psi_1|^2 dx + \frac{\lambda_1 - \delta}{n+\alpha} \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn}} dx, \tag{11}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g(u) \nabla u \nabla \psi_1 dx &= \frac{kn+1}{n} \int_{\Omega} \frac{u^n}{\psi_1^{kn+2}} g(u) |\nabla \psi_1|^2 dx + \frac{\lambda_1 - \delta}{n} \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx \\
&\quad - \frac{1}{n} \int_{\Omega} \frac{u^n}{\psi_1^{kn+1}} g'(u) \nabla u \nabla \psi_1 dx. \tag{12}
\end{aligned}$$

Combining the sixth term and the seventh term in (10) with (12) yields

$$\begin{aligned}
& -n \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx - k[n(n-2) - 2c_0^{-2}] \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g(u) \nabla u \nabla \psi_1 dx \\
& = -\frac{k(kn+1)}{n} [n(n-2) - 2c_0^{-2}] \int_{\Omega} \frac{u^n}{\psi_1^{kn+2}} g(u) |\nabla \psi_1|^2 dx \\
& \quad - \frac{k(\lambda_1 - \delta)}{n} [n(n-2) - 2c_0^{-2}] \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx - \left(2 + \frac{2c_0^{-2}}{n}\right) \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx \\
& \quad - \frac{n(n-2) - 2c_0^{-2}}{n} \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g'(u) \nabla u (\psi \nabla u - ku \nabla \psi) dx. \tag{13}
\end{aligned}$$

The last term of (13) can be estimated by

$$\begin{aligned}
& \frac{n(n-2) - 2c_0^{-2}}{n} \left| \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g'(u) \nabla u (\psi \nabla u - ku \nabla \psi) dx \right| \\
& \leq (n-2) \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+1}} g'(u) |\nabla u| |\psi \nabla u - ku \nabla \psi| dx \\
& \leq c_0 [n(n-1) - c_0^{-2}] \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn+2}} g'(u) |\psi_1 \nabla u - ku \nabla \psi_1|^2 dx \\
& \quad + \frac{(n-2)^2}{c_0 [n(n-1) - c_0^{-2}]} \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx. \tag{14}
\end{aligned}$$

Substituting (11), (13) and (14) into (10) and using (5), we find

$$\begin{aligned}
w'_n(t) &\leq -\frac{kn}{n+\alpha} [(n-2+2\alpha-2\varepsilon)(kn+1) - (n+\alpha)(n-1+\alpha-\varepsilon)k] \\
&\quad \times \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+2}} |\nabla \psi_1|^2 dx \\
&\quad - \frac{n}{n+\alpha} [k(n-2+2\alpha-2\varepsilon)(\lambda_1-\delta) - (n+\alpha)(b+\varepsilon)] \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn}} dx \\
&\quad - [n(n-1) - c_0^{-2}] \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn+2}} [g(u) - c_0 u g'(u)] |\psi_1 \nabla u - ku \nabla \psi_1|^2 dx \\
&\quad - \frac{k}{n} [(n^2 - 2n - 2c_0^{-2})(kn+1) - kn^2(n-1) + kn c_0^{-2}] \int_{\Omega} \frac{u^n}{\psi_1^{kn+2}} g(u) |\nabla \psi_1|^2 dx \\
&\quad - \left[2 + \frac{2c_0^{-2}}{n} - \frac{(n-2)^2}{c_0[n(n-1) - c_0^{-2}]} \right] \int_{\Omega} \frac{u^{n-1}}{\psi_1^{kn}} g'(u) |\nabla u|^2 dx \\
&\quad - c_0^{-2} \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn}} g(u) |\nabla u|^2 dx + nc(\varepsilon) \int_{\Omega} \frac{u^{n-1-\sigma}}{\psi_1^{kn}} dx \\
&\quad - \frac{k(\lambda_1-\delta)}{n} [n(n-2) - 2c_0^{-2}] \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx \\
&\leq -\frac{kn}{n+\alpha} [(1-k-\varepsilon k)n + (k\alpha-2)(1-\alpha+\varepsilon)] \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn+2}} |\nabla \psi_1|^2 dx \\
&\quad - \frac{n}{n+\alpha} [(\lambda_1 k - b - \delta k - \varepsilon)n - 2k(\lambda_1 - \delta)(1 - \alpha + \varepsilon) - \alpha(b + \varepsilon)] \int_{\Omega} \frac{u^{n+\alpha}}{\psi_1^{kn}} dx \\
&\quad - k \left[n(1-k) - c_0^{-2} k - 2 - \frac{2}{nc_0^2} \right] \int_{\Omega} \frac{u^n}{\psi_1^{kn+2}} g(u) |\nabla \psi_1|^2 dx + nc(\varepsilon) \int_{\Omega} \frac{u^{n-1-\sigma}}{\psi_1^{kn}} dx \\
&\quad - c_0^{-2} \int_{\Omega} \frac{u^{n-2}}{\psi_1^{kn}} \left[g(u) - \left(\frac{(n-2)^2}{c_0[n(n-1) - c_0^{-2}]} - 2 - \frac{2}{nc_0^2} \right) c_0^2 u g'(u) \right] |\nabla u|^2 dx \\
&\quad - k \left[(\lambda_1 - \delta)(n-2) - \frac{2}{nc_0^2} \right] \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx. \tag{15}
\end{aligned}$$

By Young inequality,

$$\begin{aligned}
c(\varepsilon)u^{n-1-\sigma} &= u^{n-1-\sigma} \left(\frac{c_0 k \lambda_1}{3} \right)^{\frac{n-1-\sigma}{n+\beta}} \cdot c(\varepsilon) \left(\frac{3}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{n+\beta}} \\
&\leq \frac{c_0 k \lambda_1}{3} u^{n+\beta} + [c(\varepsilon)]^{\frac{n+\beta}{1+\beta+\sigma}} \left(\frac{3}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{1+\beta+\sigma}}, \tag{16}
\end{aligned}$$

which implies that, by (5)

$$\begin{aligned} nc(\varepsilon) \int_{\Omega} \frac{u^{n-1-\sigma}}{\psi_1^{kn}} dx &\leq \frac{nc_0 k \lambda_1}{3} \int_{\Omega} \frac{u^{n+\beta}}{\psi_1^{kn}} dx + n[c(\varepsilon)]^{\frac{n+\beta}{1+\beta+\sigma}} \left(\frac{3}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{1+\beta+\sigma}} \int_{\Omega} \frac{1}{\psi_1^{kn}} dx \\ &\leq \frac{nk\lambda_1}{3} \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx + n[c(\varepsilon)]^{\frac{n+\beta}{1+\beta+\sigma}} \left(\frac{3}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{1+\beta+\sigma}} \int_{\Omega} \frac{1}{\psi_1^{kn}} dx. \end{aligned} \quad (17)$$

Similarly, if $\beta > 0$, we have (note that the following term is not shown in (15), but its negative term can be used for boundedness in (19) below)

$$n \int_{\Omega} \frac{u^n}{\psi_1^{kn}} dx \leq \frac{nk\lambda_1}{3} \int_{\Omega} \frac{u^n}{\psi_1^{kn}} g(u) dx + n \left(\frac{3}{c_0 k \lambda_1} \right)^{\frac{n}{\beta}} \int_{\Omega} \frac{1}{\psi_1^{kn}} dx. \quad (18)$$

Choosing n sufficiently large and ε sufficiently small, substituting (17) and (18) into (15) and using (5), we obtain

$$w'_n(t) \leq -n \int_{\Omega} \frac{u^n}{\psi_1^{kn}} dx + n \left[[c(\varepsilon)]^{\frac{n+\beta}{1+\beta+\sigma}} \left(\frac{2}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{1+\beta+\sigma}} + \left(\frac{2}{c_0 k \lambda_1} \right)^{\frac{n}{\beta}} \right] \int_{\Omega} \frac{1}{\psi_1^{kn}} dx. \quad (19)$$

or

$$w_n(t) \leq w_n(0) \exp(-nt) + 2[c(\varepsilon)]^{\frac{n+\beta}{1+\beta+\sigma}} \left(\frac{2}{c_0 k \lambda_1} \right)^{\frac{n-1-\sigma}{1+\beta+\sigma}} \int_{\Omega} \frac{1}{\psi_1^{kn}} dx.$$

Taking n^{th} roots and letting $n \rightarrow \infty$, we have

$$\max_{\bar{\Omega}} \frac{u(x,t)}{\psi_1^k(x)} \leq \max_{\bar{\Omega}} \frac{\phi(x)}{\psi_1^k(x)} \exp(-t) + \frac{1}{\min_{\bar{\Omega}} \psi_1^k(x)} \left(\frac{2c(\varepsilon)}{c_0 k \lambda_1} \right)^{\frac{1}{1+\beta+\sigma}}, \quad (20)$$

for all t such that the solution exists. Hence the solution is bounded for all time. $\blacksquare$

Theorem 2 *Assume that*

$$\alpha g(u) \geq ug'(u) \geq 0, \quad \text{and} \quad 0 \leq c_1 \lambda_1 ug(u) \leq f(u, \nabla u), \quad (21)$$

for any $u \in C^1(\Omega)$, $u \geq 0$ and some $c_1 > 1$, and assume that for any $\varepsilon > 0$, there exists a constant $N > 1$ such that

$$\varepsilon u + G(u) \geq 0, \quad \text{where} \quad G(u) \triangleq \int_N^u g(u) u^{-\alpha} du, \quad (22)$$

for all $u \geq N$. If $b > \lambda_1$ and $\alpha > 1$, then, for any initial value $\phi(x) \in C_0^{1+\beta}(\overline{\Omega})$, the solution of the problem (1) and (2) must blow up in a finite time.

Proof. For any positive number m , define

$$z_m(t) = \int_{\Omega} \frac{\psi^{km}(x)}{u^m(x, t)} dx, \quad (23)$$

where k satisfies $c_1 > k > 1$ and $b > \lambda_1 k$. Then

$$\begin{aligned} \frac{d}{dt} z_m(t) &= -m \int_{\Omega} \frac{\psi^{km}}{u^{m+1}} \{ [u^{\alpha} + g(u)] \Delta u + bu^{\alpha+1} + f(u, \nabla u) \} dx \\ &= -m(m+1-\alpha) \int_{\Omega} \frac{\psi^{km}}{u^{m+2-\alpha}} |\nabla u|^2 dx + km^2 \int_{\Omega} \frac{\psi^{km-1}}{u^{m+1-\alpha}} \nabla u \nabla \psi dx \\ &\quad - m(m+1) \int_{\Omega} \frac{\psi^{km}}{u^{m+2}} g(u) |\nabla u|^2 dx + m \int_{\Omega} \frac{\psi^{km}}{u^{m+1}} g'(u) |\nabla u|^2 dx \\ &\quad + km^2 \int_{\Omega} \frac{\psi^{km-1}}{u^{m+1}} g(u) \nabla u \nabla \psi dx - mb \int_{\Omega} \frac{\psi^{km}}{u^{m-\alpha}} dx \\ &\quad - m \int_{\Omega} \frac{\psi^{km}}{u^{m+1}} f(u, \nabla u) dx. \end{aligned} \quad (24)$$

Using (9), we have

$$\begin{aligned} \frac{d}{dt} z_m(t) &= -m(m+1-\alpha) \int_{\Omega} \frac{\psi^{km-2}}{u^{m+2-\alpha}} |\psi \nabla u - ku \nabla \psi|^2 dx \\ &\quad - km(m+2-2\alpha) \int_{\Omega} \frac{\psi^{km-1}}{u^{m+1-\alpha}} \nabla u \nabla \psi dx \\ &\quad + m(m+1-\alpha) k^2 \int_{\Omega} \frac{\psi^{km-2}}{u^{m-\alpha}} |\nabla \psi|^2 dx \\ &\quad - m \int_{\Omega} \frac{\psi^{km-2}}{u^{m+2}} [(m+1)g(u) - ug'(u)] |\psi \nabla u - ku \nabla \psi|^2 dx \\ &\quad - km(m+2) \int_{\Omega} \frac{\psi^{km-1}}{u^{m+1}} g(u) \nabla u \nabla \psi dx + 2km \int_{\Omega} \frac{\psi^{km-1}}{u^m} g'(u) \nabla u \nabla \psi dx \\ &\quad + mk^2 \int_{\Omega} \frac{\psi^{km-2}}{u^m} [(m+1)g(u) - ug'(u)] |\nabla \psi|^2 dx \\ &\quad - mb \int_{\Omega} \frac{\psi^{km}}{u^{m-\alpha}} dx - m \int_{\Omega} \frac{\psi^{km}}{u^{m+1}} f(u, \nabla u) dx. \end{aligned} \quad (25)$$

Integration by parts for the mixed gradient term yields

$$\int_{\Omega} \frac{\psi^{km-1}}{u^{m+1-\alpha}} \nabla u \nabla \psi dx = \frac{km-1}{m-\alpha} \int_{\Omega} \frac{\psi^{km-2}}{u^{m-\alpha}} |\nabla \psi|^2 dx - \frac{\lambda_1}{m-\alpha} \int_{\Omega} \frac{\psi^{km}}{u^{m-\alpha}} dx, \quad (26)$$

and

$$\begin{aligned} & -km(m+2) \int_{\Omega} \frac{\psi^{km-1}}{u^{m+1}} g(u) \nabla u \nabla \psi dx + 2km \int_{\Omega} \frac{\psi^{km-1}}{u^m} g'(u) \nabla u \nabla \psi dx \\ & = -k(m+2)(km-1) \int_{\Omega} \frac{\psi^{km-2}}{u^m} g(u) |\nabla \psi|^2 dx \\ & \quad + k(m-2) \int_{\Omega} \frac{\psi^{km-1}}{u^m} g'(u) \nabla u \nabla \psi dx \\ & \quad + k(m+2)\lambda_1 \int_{\Omega} \frac{\psi^{km}}{u^m} g(u) dx. \end{aligned} \quad (27)$$

The mixed gradient term in (27) can be estimated by

$$\begin{aligned} \int_{\Omega} \frac{\psi^{km-1}}{u^m} g'(u) \nabla u \nabla \psi dx & = \int_{\Omega} \frac{\psi^{km-2}}{u^m} g'(u) (\psi \nabla u - ku \nabla \psi) \nabla \psi dx \\ & \quad + k \int_{\Omega} \frac{\psi^{km-2}}{u^{m-1}} g'(u) |\nabla \psi|^2 dx \\ & \leq \frac{m}{2k\alpha} \int_{\Omega} \frac{\psi^{km-2}}{u^{m+1}} g'(u) |\psi \nabla u - ku \nabla \psi|^2 dx \\ & \quad + \left(\frac{k\alpha}{2m} + k \right) \int_{\Omega} \frac{\psi^{km-2}}{u^{m-1}} g'(u) |\nabla \psi|^2 dx. \end{aligned} \quad (28)$$

Substituting (26), (27) and (28) into (25) and using (21), we obtain

$$\begin{aligned} \frac{d}{dt} z_m(t) & \leq -\frac{mk}{m-\alpha} [(m+2-2\alpha)(km-1) - k(m+1-\alpha)(m-\alpha)] \int_{\Omega} \frac{\psi^{km-2}}{u^{m-\alpha}} |\nabla \psi|^2 dx \\ & \quad + \left[\frac{\lambda_1}{m-\alpha} mk(m+2-2\alpha) - mb \right] \int_{\Omega} \frac{\psi^{km}}{u^{m-\alpha}} dx \\ & \quad - m \int_{\Omega} \frac{\psi^{km-2}}{u^{m+2}} \left[(m+1)g(u) - \left(\frac{m-2}{2\alpha} + 1 \right) ug'(u) \right] |\psi \nabla u - ku \nabla \psi|^2 dx \\ & \quad - k[(m+2)(km-1) - km(m+1)] \int_{\Omega} \frac{\psi^{km-2}}{u^m} g(u) |\nabla \psi|^2 dx \\ & \quad - \left[mk^2 - k(m-2) \left(\frac{k\alpha}{2m} + k \right) \right] \int_{\Omega} \frac{\psi^{km-2}}{u^{m-1}} g'(u) |\nabla \psi|^2 dx \\ & \quad + k(m+2)\lambda_1 \int_{\Omega} \frac{\psi^{km}}{u^m} g(u) dx - m \int_{\Omega} \frac{\psi^{km}}{u^{m+1}} f(u, \nabla u) dx. \end{aligned} \quad (29)$$

Choosing m sufficiently large, we find

$$\begin{aligned}
z'_m(t) &\leq -\frac{mk}{m-\alpha} [(k-1)m + (k\alpha - 2)(1-\alpha)] \int_{\Omega} \frac{\psi^{mk-2}}{u^{m-\alpha}} |\nabla \psi|^2 dx \\
&\quad - \frac{m}{m-\alpha} [(b - k\lambda_1)m - 2k\lambda_1(1-\alpha) - \alpha b] \int_{\Omega} \frac{\psi^{mk}}{u^{m-\alpha}} dx \\
&\quad - k[(k-1)m - 2] \int_{\Omega} \frac{\psi^{mk-2}}{u^m} g(u) |\nabla \psi|^2 dx \\
&\quad - k^2 \left[2 - \frac{\alpha(m-2)}{2m} \right] \int_{\Omega} \frac{\psi^{mk-2}}{u^{m-1}} g'(u) |\nabla \psi|^2 dx \\
&\quad - \int_{\Omega} \frac{\psi^{mk}}{u^{m+1}} [mf(u, \nabla u) - (m+2)k\lambda_1 ug(u)] dx \\
&\leq -k \int_{\Omega} \frac{\psi^{mk-2}}{u^m} \left\{ [(k-1)m - 2]g(u) + k \left[2 - \frac{\alpha(m-2)}{2m} \right] ug'(u) \right\} |\nabla \psi|^2 dx \\
&\leq 0.
\end{aligned} \tag{30}$$

Thus, we obtain $z_m(t) \leq z_m(0)$. Taking m^{th} roots and letting $m \rightarrow \infty$, we have

$$\max_{\Omega} \frac{\psi^k(x)}{u(x, t)} \leq \max_{\Omega} \frac{\psi^k(x)}{\phi(x)}, \tag{31}$$

for all t such that the solution exists and

$$v(t) = \sup_{\Omega} \frac{\psi^k(x)}{u(x, t)}$$

is decreasing.

To prove that $u(x, t)$ blows up in a finite time we need to show that the maximal existence time $T^* < \infty$. Suppose instead that $T^* = \infty$ and $\lim_{t \rightarrow \infty} v(t) = a$. If $a > 0$, then $u(x, t)$ is bounded in any domain D_1 with $\overline{D_1} \subset \Omega$. It follows from (30) that

$$z'_m(t) \leq -m\delta_0 \int_{\Omega} \frac{\psi^{km}}{u^m} u^{\alpha} dx \leq -m\delta_0 \int_{D_1} \frac{\psi^{km}}{u^m} u^{\alpha} dx \leq -c_m \tag{32}$$

for sufficiently large but fixed m . Then we get a contradiction because t cannot approach

infinity. If $a = 0$, then for any $N > 1$,

$$\frac{\psi^k(x)}{u(x, t_1)} \leq v(t_1) < \frac{1}{N} \quad \text{or} \quad u(x, t_1) \geq N\psi^k(x), \quad (33)$$

in D_1 , for sufficiently large t_1 and (33) holds for all $t \geq t_1$. By the continuity of the eigenvalues with respect to the domain Ω (see [17]), we can find a small $\delta > 0$ such that $b > \lambda_1 + \delta$ and we can define a domain $D_1 \subset \Omega$ such that ∂D_1 is located inside of Ω and there exists the first normalized eigenfunction of

$$\Delta\psi_2 + (\lambda_1 + \delta)\psi_2 = 0 \quad (34)$$

in D_1 with $\psi_2|_{\partial D_1} = 0$. Then u is positive in $\overline{D_1}$. Let

$$y(t) = \int_{D_1} \frac{\psi_2^k}{u^{\alpha-1}} dx, \quad (35)$$

where $k > 1$ and $b > (\lambda_1 + \delta)k$. Then, using (21)

$$\begin{aligned} y'(t) &= -(\alpha - 1) \int_{D_1} \frac{\psi_2^k}{u^\alpha} \{ [u^\alpha + g(u)] \Delta u + bu^{\alpha+1} + f(u, \nabla u) \} dx \\ &= k(\alpha - 1) \int_{D_1} \psi_2^{k-1} [1 + g(u)u^{-\alpha}] \nabla \psi_2 \nabla u dx \\ &\quad - (\alpha - 1) \int_{D_1} \frac{\psi_2^k}{u^{\alpha+1}} [\alpha g(u) - ug'(u)] |\nabla u|^2 dx - b(\alpha - 1) \int_{D_1} \psi_2^k u dx \\ &\quad - (\alpha - 1) \int_{D_1} \frac{\psi_2^k}{u^\alpha} f(u, \nabla u) dx \\ &\leq -k(\alpha - 1) \int_{D_1} \psi_2^{k-1} [u + G(u)] \Delta \psi_2 dx - b(\alpha - 1) \int_{D_1} \psi_2^k u dx \\ &\quad - k(k-1)(\alpha - 1) \int_{D_1} \psi_2^{k-2} [u + G(u)] |\nabla \psi_2|^2 dx \\ &= -\frac{1}{2}(\alpha - 1) [b - k(\lambda_1 + \delta)] \int_{D_1} \psi_2^k u dx \\ &\quad - k(\alpha - 1)(\lambda_1 + \delta) \int_{D_1} \psi_2^k \left[\frac{b - k(\lambda_1 + \delta)}{2k(\lambda_1 + \delta)} u - G(u) \right] dx \\ &\quad - k(k-1)(\alpha - 1) \int_{D_1} \psi_2^{k-2} [u + G(u)] |\nabla \psi_2|^2 dx. \end{aligned} \quad (36)$$

By (22) and (33), we can find an $N > 1$, such that

$$\begin{aligned} y'(t) &\leq -\frac{1}{2}(\alpha - 1)[b - k(\lambda_1 + \delta)] \int_{D_1} \psi_2^k u dx \\ &\leq -\frac{1}{2}(\alpha - 1)[b - k(\lambda_1 + \delta)] N \int_{D_1} \psi_2^k \psi^k dx, \end{aligned} \quad (37)$$

for all $u > N$. Since $y(t)$ is positive, (37) doesn't hold for all $t > 0$. Hence, $u(x, t)$ must blow up in a finite time. ■

Example. If $\alpha = 3$, $b = 1$, $g(u) = u^2$ and $f(u, \nabla u) = \gamma u^3 + u|\nabla u|^\beta$ with $\gamma > 1$ and $\beta < 2$, then all conditions in Theorem 1 and Theorem 2 are satisfied. When $\lambda_1 > 1$ all solutions are bounded, while $\lambda_1 < 1$ all solutions blow up in a finite time.

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